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Excitation of instability waves in a two-dimensional shear layer by sound

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The excitation of instability waves in a plane compressible shear layer by sound waves is studied. The problem is formulated mathematically as an inhomogeneous boundary-value problem. A general solution for arbitrary incident sound wave is found by first constructing the Green's function of the problem. Numerical values of the coupling constants between incident sound waves and excited instability waves for a range of flow Mach numbers are calculated. The effect of the angle of incidence in the case of a beam of acoustic waves is analysed. It is found that for moderate subsonic Mach numbers a narrow beam aiming at an angle between 50 to 80° to the flow direction is most effective in exciting instability waves.

1. Introduction

This paper considers the excitation of instability waves of a two-dimensional compressible shear layer by incident sound waves. The flow configuration is illustrated in figure 1. In the past, sound waves of discrete frequencies have been used by many experimenters to excite flow instabilities in transition studies. Miksad (1973) employed an experimental set-up almost identical to that of figure 1 in his investigation of non-linear instability waves in plane shear layer. Freymuth (1966) used sound waves of discrete frequency to induce unstable waves in the mixing layer of a jet in his study of inviscid linear instability wave characteristics. Sato (1970) used sound to trigger unstable waves in two-dimensional wakes in his wake transition experiments. In addition to these intentional uses of sound as an exciter in laboratory flow transition experiments, knowledge of how incident sound waves induce flow instabilities has obvious practical applications in wind tunnel measurements and laminar flow control in long range aircraft technology.

In a recent review article on boundary-layer stability and transition, Reshotko (1976) briefly discussed the problem of the response of a boundary layer to a moving sound wave. It was pointed out that receptivity phenomena of this kind differed from the usual stability problem both physically and mathematically. The mathematical problem involved can no longer be formulated as a normal mode or an eigenvalue problem. Although some initial work has been done by Mack (1975) yet a satisfactory mathematical formulation of the problem does not seem to be available. Mack's work was motivated by the experimental observations of Kendall (1975) on the growth of supersonic boundary-layer instability waves. In this work he only treated the response

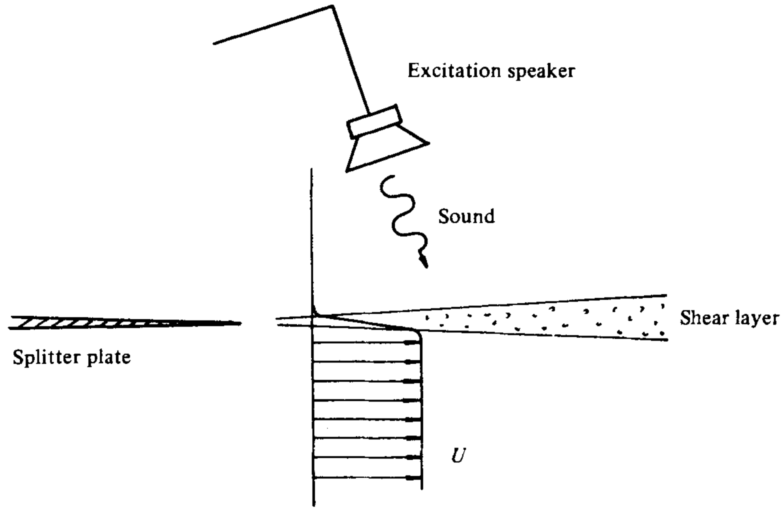

FIGURE 1. Flow configuration under consideration.

of a stable boundary layer to incident sound. A neutral stable solution of the Orr–Sommerfeld equation in the boundary layer was used to match an incoming acoustic wave solution with a prescribed amplitude and a reflected wave solution at the outer edge of the layer. In this way the forced disturbance amplitude was determined. Mack used this solution up to the point where the laminar boundary layer became unstable. Downstream of the neutral stable point he assumed that the boundary-layer disturbance was given by the usual instability wave solution. Acoustic forcing of unstable waves was ignored.

The objective of this paper is to study the effects of incident sound waves on an unstable shear layer. In other words, we are dealing with the forced excitation of unstable waves. In § 2, this problem is formulated mathematically as an inhomogeneous boundary-value problem. A general solution is constructed by using an appropriate Green's function. The advantage of this approach is that arbitrary spatial distribution of the incident sound wave amplitude is automatically taken care of. For a plane shear layer at moderately high Reynolds number, it has been shown by Michalke (1965) and Freymuth (1966) that viscosity is not important as far as the instability characteristics of the layer is concerned. Accordingly, we shall use an inviscid model in our analysis. Solution of the problem is provided in § 3. It turns out that the coupling constants between the incident sound wave and the excited instability wave for a given mean flow profile depend on the frequency and the flow Mach number alone. These constants can, therefore, be computed once and for all. For a beam of incident sound waves the angle of incidence is an important parameter. The effect of incident angle is studied in § 4 in relation to acoustic beams with Gaussian amplitude distributions. It is found that for moderate subsonic Mach numbers an acoustic beam inclined at an angle between 50 and 80° to the flow direction is most effective in exciting the instability waves. In addition, the calculation indicates that the beam width is also an important factor in determining the amplitude of the excited instability wave. To induce flow instabilities a narrow beam is preferred.

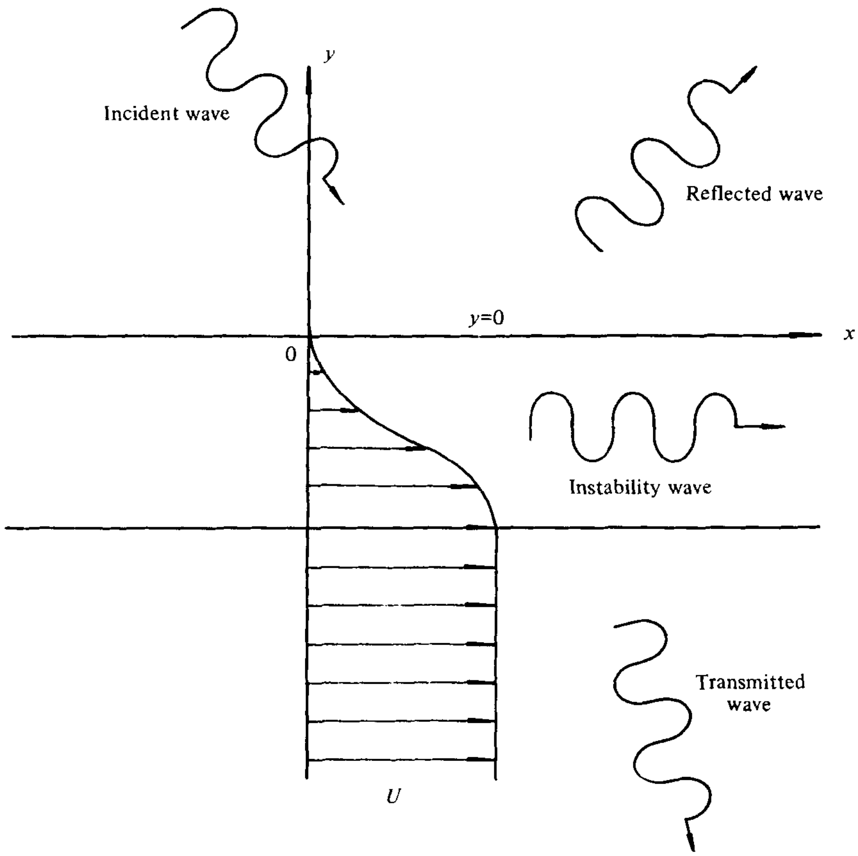

FIGURE 2. Schematic diagram showing incident, reflected and transmitted waves together with co-ordinate system used.

2. Formulation

Consider a beam of sound waves of frequency Ω incident on a two-dimensional shear layer as shown in figure 2. Part of the sound waves would be reflected by the shear layer and part of it would be transmitted. If the frequency of oscillation Ω lies in the unstable frequency range of the mixing layer then instability waves would also be excited. To facilitate the formulation of this problem mathematically, we shall assume that the mean flow is parallel at least locally and the fluid is compressible and inviscid. The parallel flow approximation is not new. It is used in most hydrodynamic stability calculations. The inviscid assumption is reasonable for moderately large Reynolds number flow. Experimentally, it has been found by Michalke (1965) and Freymuth (1966) to be valid for stability consideration. Recent experiments by Chan (1974*a, b*) and Moore (1977) seem to indicate that this is true even if the shear layer is turbulent.

Above the shear layer, i.e. for $y \geqslant 0$ (see figure 2), there is no mean flow or practically no mean flow. In this region small amplitude disturbances are given by solutions of the simple wave equation. If p_i and v_i denote respectively the pressure and the velocity component in the y direction associated with the incident acoustic wave, and p_r and v_r those associated with the reflected wave then we have

$$\nabla^2 p_i - \frac{1}{a_0^2}\frac{\partial^2 p_i}{\partial t^2} = 0, \quad \nabla^2 p_r - \frac{1}{a_0^2}\frac{\partial^2 p_r}{\partial t^2} = 0, \tag{1}$$

and

$$\rho_0 \frac{\partial v_i}{\partial t} = -\frac{\partial p_i}{\partial y}, \quad \rho_0 \frac{\partial v_r}{\partial t} = -\frac{\partial p_r}{\partial y}. \tag{2}$$

Equation (2) is the linearized y momentum equation. a_0 is the speed of sound in the ambient fluid. ρ_0 is the density. We shall consider incident sound waves with time dependence of the form $\exp(-i\Omega t)$. By applying the Fourier transform to (1) and (2) we show that p_i and v_i are given by

$$p_i(x,y,t) \equiv \hat{p}_i(x,y)\exp(-i\Omega t)$$
$$= \int_{-\infty}^{\infty} g(k)\exp\left[-i((\Omega/a_0)^2-k^2)^{\frac{1}{2}} y + ikx - i\Omega t\right] dk, \tag{3}$$

$$v_i(x,y,t) \equiv \hat{v}_i(x,y)\exp(-i\Omega t)$$
$$= \frac{-1}{\rho_0 \Omega}\int_{-\infty}^{\infty} g(k)\left[(\Omega/a_0)^2-k^2\right]^{\frac{1}{2}}\exp\left[-i((\Omega/a_0)^2-k^2)^{\frac{1}{2}} y + ikx - i\Omega t\right] dk. \tag{4}$$

The branch of the square root to be used is

$$\left.\begin{aligned} &\operatorname{Re}((\Omega/a_0)^2-k^2)^{\frac{1}{2}} > 0, \quad k^2 < (\Omega/a_0)^2, \\ &\operatorname{Im}((\Omega/a_0)^2-k^2)^{\frac{1}{2}} > 0, \quad k^2 > (\Omega/a_0)^2. \end{aligned}\right\} \tag{5}$$

In (3) and (4), the amplitude and direction of propagation of the incident acoustic wave is determined by the function $g(k)$.

For $y \leqslant 0$, i.e. inside and below the shear layer, disturbances must satisfy the linearized momentum and energy equations. Let $\bar{u}$, $\bar{\rho}$, $\bar{p} = p_0$ (a constant) and γ denote the mean flow velocity (in the x direction), density, pressure and ratio of specific heats respectively, then these equations are

$$\bar{\rho}\left(\frac{\partial u}{\partial t} + \bar{u}\frac{\partial u}{\partial x} + v\frac{d\bar{u}}{dy}\right) = -\frac{\partial p}{\partial x}, \tag{6}$$

$$\bar{\rho}\left(\frac{\partial v}{\partial t} + \bar{u}\frac{\partial v}{\partial x}\right) = -\frac{\partial p}{\partial y}, \tag{7}$$

$$\frac{\partial p}{\partial t} + \bar{u}\frac{\partial p}{\partial x} + \gamma p_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{8}$$

where (u, v) and p are the disturbance velocity and pressure. By eliminating u and v a single equation in terms of p can be formed:

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)^3 p - \frac{\gamma p_0}{\bar{\rho}}\left[\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) - \frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{dy}\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\frac{\partial p}{\partial y} - 2\frac{d\bar{u}}{dy}\frac{\partial^2 p}{\partial x \partial y}\right] = 0. \tag{9}$$

The boundary conditions of the present problem are:

As $y \to \infty$, p_r satisfies the radiation and boundedness condition. (10)

At $y = 0$,

$$\hat{p}_i(x,0)\exp(-i\Omega t) + p_r(x,0,t) = p(x,0,t), \tag{11}$$

$$\hat{v}_i(x,0)\exp(-i\Omega t)+v_r(x,0,t) = v(x,0,t). \tag{12}$$

As $y \to -\infty$, p satisfies the radiation and boundedness condition. (13)

Equations (1)–(13) form an inhomogeneous boundary-value problem. Here $\hat{p}_i(x,0)$ and $\hat{v}_i(x,0)$ in (11) and (12) are arbitrary given input functions. For completely general incident sound waves it is found that the problem can best be solved by first developing an appropriate Green's function. The Green's function satisfies the same problem as above except that the inhomogeneous terms of boundary conditions (11) and (12) are replaced by delta functions as follows:

$$\alpha\,\delta(x-\xi)\exp(-i\Omega t)+p_r(x,0,t) = p(x,0,t), \tag{14}$$

$$\beta\,\delta(x-\xi)\exp(-i\Omega t)+v_r(x,0,t) = v(x,0,t). \tag{15}$$

Let $p_\alpha(\xi;x,y,t)$ be the solution of p with $\beta = 0$ and $\alpha = 1$ and $p_\beta(\xi;x,y,t)$ be the solution of p with $\beta = 1$ and $\alpha = 0$. It is easy to verify that the solution of the general problem satisfying boundary conditions (11) and (12) is

$$p(x,y,t) = \int_{-\infty}^{\infty}\hat{p}_i(\xi,0)\,p_\alpha(\xi;x,y,t)\,d\xi+\int_{-\infty}^{\infty}\hat{v}_i(\xi,0)\,p_\beta(\xi;x,y,t)\,d\xi. \tag{16}$$

The unstable wave solution is included in (16). The task of determining the excited instability wave, therefore, reduces to that of constructing $p_\alpha(\xi;x,y,t)$ and $p_\beta(\xi;x,y,t)$. This is carried out in the next section.

3. Excited instability wave solution

We shall denote the Fourier–Laplace transform of a function $f(x,t)$ by $\tilde{f}(k,\omega)$. These functions are related by

$$\left.\begin{aligned} \tilde{f}(k,\omega) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(x,t)\exp(-ikx+i\omega t)\,dt\,dx,\\ f(x,t) &= \int_{\Gamma}\int_{-\infty}^{\infty}\tilde{f}(k,\omega)\exp(ikx-i\omega t)\,dk\,d\omega, \end{aligned}\right\} \tag{17}$$

where Γ is a contour parallel to the real axis in the ω plane above all singularities. On applying the Fourier–Laplace transform to (1) and (2) we find $\tilde{p}_r$ and $\tilde{v}_r$ are solutions of the following equations:

$$\frac{d^2\tilde{p}_r}{dy^2}-\left(k^2-\frac{\omega^2}{a_0^2}\right)\tilde{p}_r = 0,$$

$$i\rho_0\omega\tilde{v}_r = \frac{d\tilde{p}_r}{dy}.$$

A solution of these equations which satisfies the radiation condition is

$$\tilde{p}_r = A\exp[-(k^2-\omega^2/a_0^2)^{\frac{1}{2}}y], \tag{18a}$$

$$\hat{v}_r = \frac{i(k^2-\omega^2/a_0^2)^{\frac{1}{2}}}{\rho_0\omega}A\exp[-(k^2-\omega^2/a_0^2)^{\frac{1}{2}}y]. \tag{18b}$$

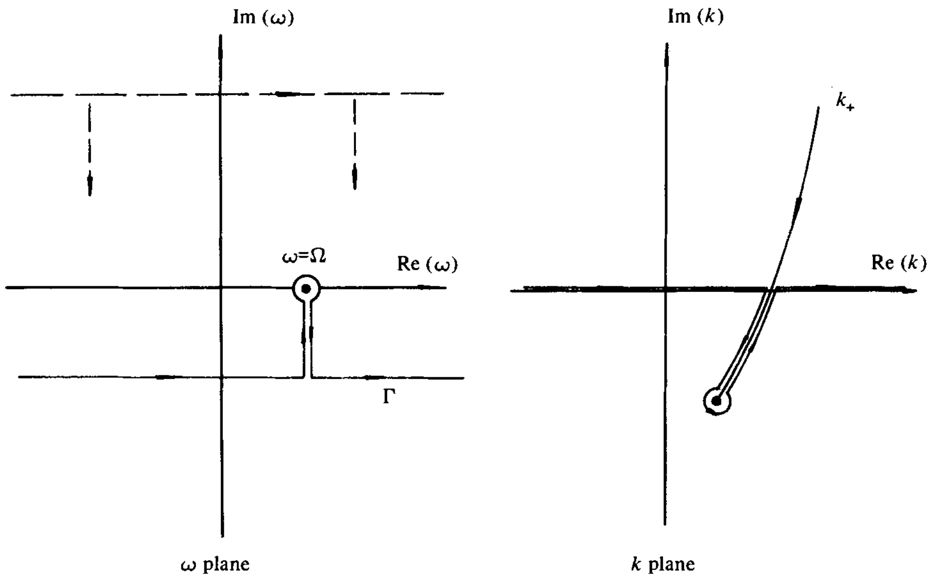

FIGURE 3. Inversion contours in the ω plane and k plane.
— → —, initial position of contour Γ.

The branch of the square root to be used is

$$\mathrm{Re}\,(k^2-\omega^2/a_0^2)^{\frac{1}{2}} > 0,$$

if $$\mathrm{Re}\,(k^2-\omega^2/a_0^2)^{\frac{1}{2}} = 0, \quad \text{use} \quad \mathrm{Im}\,(k^2-\omega^2/a_0^2)^{\frac{1}{2}} < 0. \tag{19}$$

The Fourier–Laplace transforms of (9) and (7) are

$$\frac{d^2\tilde{p}}{dy^2}+\left[\frac{2k}{(\omega-\bar{u}k)}\frac{d\bar{u}}{dy}-\frac{1}{\bar{\rho}}\frac{d\bar{\rho}}{dy}\right]\frac{d\tilde{p}}{dy}+\left[\frac{(\omega-\bar{u}k)^2}{\gamma p_0}\bar{\rho}-k^2\right]\tilde{p}=0, \tag{20}$$

$$i\bar{\rho}(\omega-\bar{u}k)\,\tilde{v}=\frac{d\tilde{p}}{dy}. \tag{21}$$

Below the shear layer the flow is uniform. Thus as $y \to -\infty$, (20) becomes

$$\frac{d^2\tilde{p}}{dy^2}-\left[k^2-\frac{(\omega-Uk)^2}{a^2}\right]\tilde{p}=0, \tag{22}$$

where U and a are the velocity and sound speed of the uniform flow. A solution of (22) which satisfies the radiation and boundedness condition (13) is

$$\tilde{p}=B\exp\left[\left(k^2-\frac{(\omega-Uk)^2}{a^2}\right)^{\frac{1}{2}}y\right], \tag{23}$$

where $$\mathrm{Re}\left[k^2-\frac{(\omega-Uk)^2}{a^2}\right]^{\frac{1}{2}} > 0;$$

if $$\mathrm{Re}\left[k^2-\frac{(\omega-Uk)^2}{a^2}\right]^{\frac{1}{2}} = 0 \quad \text{use the branch} \quad \mathrm{Im}\left[k^2-\frac{(\omega-Uk)^2}{a^2}\right]^{\frac{1}{2}} < 0.$$

We shall denote the solutions of (20) and (21), which tends to (23) for large negative values of y, by

$$\tilde{p} = B\phi(y, k, \omega), \tag{24a}$$

$$\tilde{v} = \frac{-iB}{\bar{\rho}(\omega - \bar{u}k)}\frac{d\phi}{dy}. \tag{24b}$$

The unknown constants A and B of (18) and (24) are to be found by the inhomogeneous boundary conditions (14) and (15) at $y = 0$. The Fourier–Laplace transforms of these boundary conditions at $y = 0$ are

$$\frac{i\alpha \exp(-ik\xi)}{4\pi^2(\omega - \Omega)} + \tilde{p}_r = \tilde{p}, \tag{25}$$

$$\frac{i\beta \exp(-ik\xi)}{4\pi^2(\omega - \Omega)} + \tilde{v}_r = \tilde{v}. \tag{26}$$

By substituting $\tilde{p}_r$, $\tilde{v}_r$, $\hat{p}$ and $\tilde{v}$ from (18) and (24) into (25) and (26) it is easy to find

$$B = \frac{i \exp(-ik\xi)}{4\pi^2(\omega - \Omega)\,\Delta(k, \omega)}[\alpha(k^2 - (\omega/a_0)^2)^{\frac{1}{2}} + i\beta\rho_0\omega], \tag{27}$$

where
$$\Delta(k, \omega) = (k^2 - (\omega/a_0)^2)^{\frac{1}{2}}\,\phi(0, k, \omega) + \frac{d\phi}{dy}(0, k, \omega). \tag{28}$$

On inverting the Fourier–Laplace transform as in (17) we obtain

$$p_\alpha(\xi; x, y, t) = \frac{i}{4\pi^2}\int_\Gamma\int_{-\infty}^{\infty} \frac{\exp(ik(x-\xi) - i\omega t)\,[k^2 - (\omega/a_0)^2]^{\frac{1}{2}}}{(\omega - \Omega)\,\Delta(k, \omega)}\,\phi(y, k, \omega)\,dk\,d\omega, \tag{29}$$

$$p_\beta(\xi; x, y, t) = \frac{-\rho_0}{4\pi^2}\int_\Gamma\int_{-\infty}^{\infty} \frac{\omega \exp(ik(x-\xi) - i\omega t)}{(\omega - \Omega)\,\Delta(k, \omega)}\,\phi(y, k, \omega)\,dk\,d\omega. \tag{30}$$

The integrals of (29) and (30) contain the complete response of the shear layer to the incident acoustic wave. Here we are only interested in the part of this response function which is related to the unstable wave solution. To evaluate these integrals we follow the procedure of Briggs (1964, chapter 2). This procedure has been used by the present author, Tam (1971), in connexion with the acoustic radiation from a supersonic jet due to instability of its thin shear layer. The instability wave arises from certain zeros of $\Delta(k, \omega)$ in the complex k plane. On following Briggs' procedure, the contour Γ is first put in the upper half ω plane with $\mathrm{Im}(\omega) \to \infty$. This step is equivalent to invoking the causality condition. Since ω is a value of Γ, one can solve for the roots, $k(\omega)$, of $\Delta(k, \omega_r + i\infty) = 0$ and determine the position of the poles of the integrand relative to the inversion contour in the k plane. Now deform contour Γ towards the real axis of the ω plane as shown in figure 3. As a result of this the roots $k(\omega)$ move about in the k plane. It was shown in our earlier work, Tam (1971), that the unstable pole $k_+(\omega)$ crosses the real k axis during this process of contour deformation. The same phenomenon happens here as is depicted in figure 3. The instability wave solution is given by the residue contributions of the pole $\omega = \Omega$ in the ω plane and the pole $k = k_+(\Omega)$ in the k plane where $\Delta(k_+(\Omega), \Omega) = 0$. Ignoring other contributions to the integrals and keeping only the excited instability wave solution we find, from (29) and (30), p_α and p_β as given below:

$$p_\alpha(\xi; x, y, t) = \frac{i(k_+^2 - \Omega^2/a_0^2)^{\frac{1}{2}}}{(\partial\Delta/\partial k)_\Omega} \phi(y, k_+, \Omega) \exp[ik_+(x-\xi) - i\Omega t] H(x-\xi), \quad (31)$$

$$p_\beta(\xi; x, y, t) = \frac{-\rho_0 \Omega}{(\partial\Delta/\partial k)_\Omega} \phi(y, k_+, \Omega) \exp[ik_+(x-\xi) - i\Omega t] H(x-\xi), \quad (32)$$

where $\Delta(k_+(\Omega), \Omega) = 0$ and $\left(\frac{\partial\Delta}{\partial k}\right)_\Omega = \frac{\partial\Delta(k,\omega)}{\partial k}\bigg|_{k=k_+(\Omega),\, \omega=\Omega}$,

$H(x-\xi) = \begin{cases} 0, & x < \xi \\ 1, & x > \xi \end{cases}$, is the unit step function.

By means of (16), (31) and (32), the excited instability wave solution is found. It may be written as

$$p(x, y, t) = \left[G \int_{-\infty}^{x} \hat{p}_i(\xi, 0) \exp(-ik_+\xi)\, d\xi + F \int_{-\infty}^{x} \hat{v}_i(\xi, 0) \exp(-ik_+\xi)\, d\xi\right] \times \phi(y, k_+, \Omega) \exp(ik_+ x - i\Omega t). \quad (33)$$

The coupling constants G and F are given by

$$G = \frac{i(k_+^2 - \Omega^2/a_0^2)^{\frac{1}{2}}}{(\partial\Delta/\partial k)_\Omega}, \quad (34a)$$

$$F = \frac{-\rho_0\Omega}{(\partial\Delta/\partial k)_\Omega}. \quad (34b)$$

In order to understand the physical meaning of (33), let us differentiate it with respect to x. The following equation for the spatial rate of increase of p is obtained:

$$\frac{\partial p}{\partial x} = ik_+ p + [G\hat{p}_i(x, 0) + F\hat{v}_i(x, 0)]\, \phi(y, k_+, \Omega) \exp(-i\Omega t). \quad (35)$$

The first term on the right-hand side of (35) is, of course, the usual local growth term of the instability wave. It is there even in the absence of external excitation. The second term is linearly proportional to the amplitude of the incident acoustic wave. Clearly it represents a forcing term. It is this term which is responsible for the forced excitation of unstable waves of the shear layer by sound.

The coupling constants G and F of (33) do not depend on the characteristics of the incident sound wave. They are functions of the properties of the shear layer and thus can be computed once and for all. According to the experimental measurements of Freymuth (1966), the mean velocity distribution of a two-dimensional mixing layer can be represented adequately by a hyperbolic tangent profile. With respect to the co-ordinate system shown in figure 2 we shall assume that the mean flow is given by

$$\frac{\bar{u}}{U} = 0{\cdot}5\left[1 - \tanh\left(\frac{y}{\delta} + 3\right)\right]. \quad (36)$$

In (36) δ is a measure of the shear-layer thickness. Also the outer edge of the shear layer has been taken to be at a distance 3δ from the point of maximum shear. The effect of density gradient on the stability characteristics of the flow is known to be

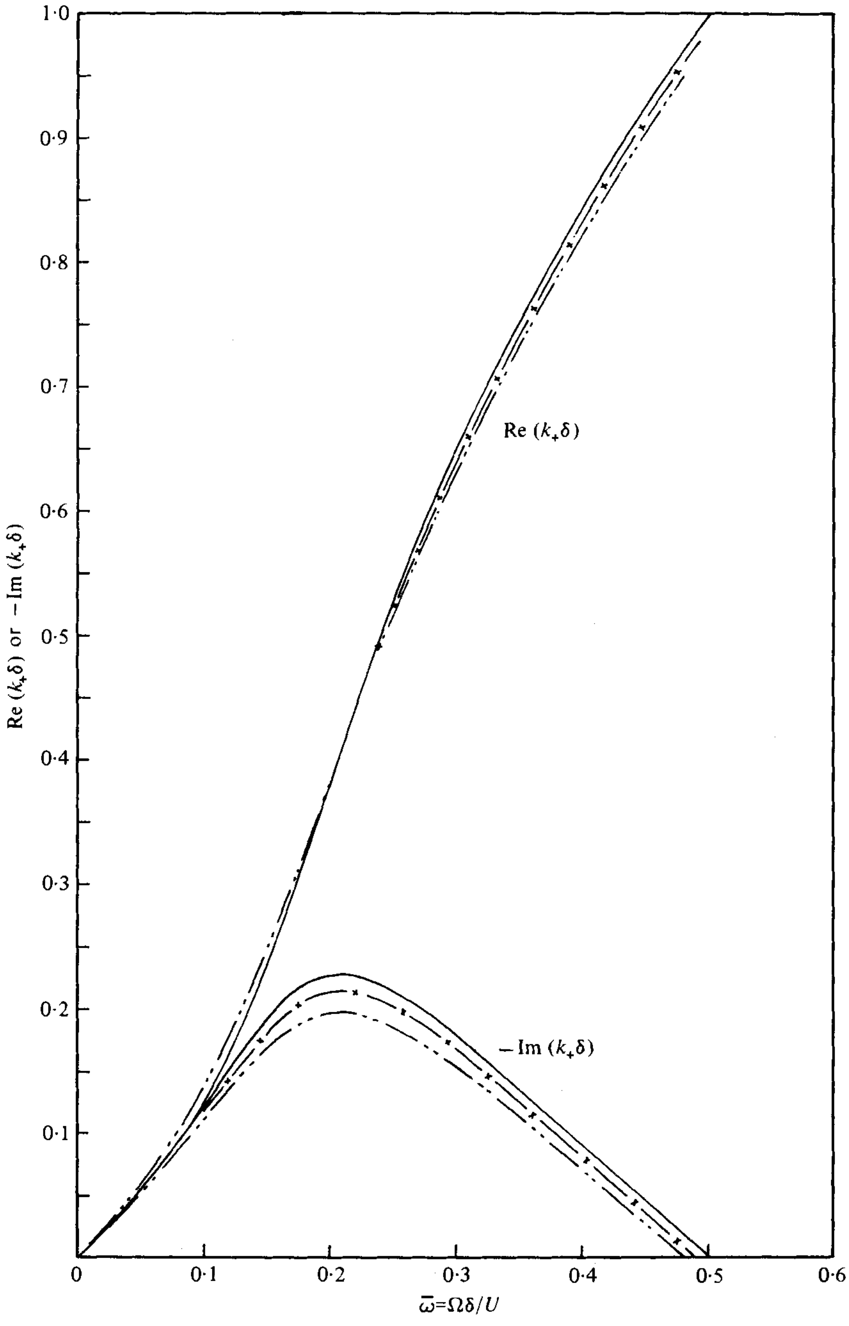

FIGURE 4. Instability characteristics of compressible plane shear layer. ——, $M = 0{\cdot}0$; ×—×, $M = 0{\cdot}4$; —··—, $M = 0{\cdot}6$.

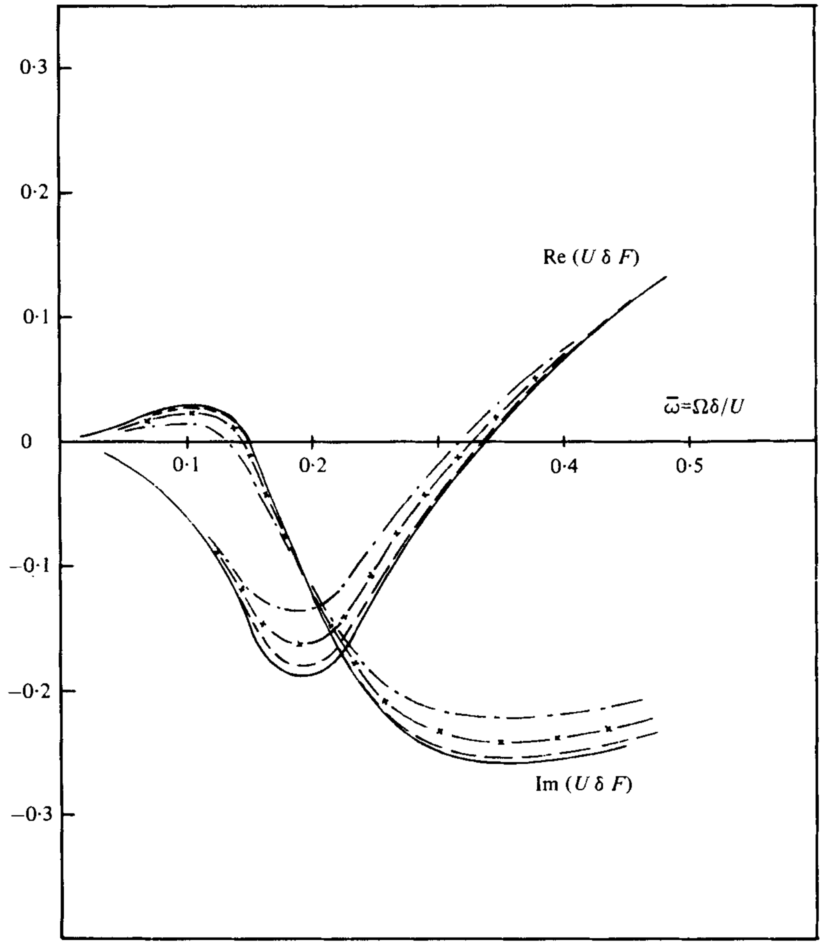

FIGURE 5. Real and imaginary parts of $U\delta F$ as functions of $\bar{\omega} = \Omega\delta/U$. ——, $M = 0{\cdot}05$; – –, $M = 0{\cdot}2$; ×—×, $M = 0{\cdot}4$; —·—, $M = 0{\cdot}6$.

small for subsonic Mach numbers. Here we shall adopt the approximation that the total temperatures of the stationary as well as the moving fluids are the same. This leads to

$$\frac{\bar{\rho}}{\rho_0} = \frac{1+\frac{1}{2}(\gamma-1)\,M^2}{1+\frac{1}{2}(\gamma-1)\,M^2[1-(\bar{u}/U)^2]}, \tag{37}$$

where $M = U/a$ is the free-stream Mach number. On using $\bar{u}$ and $\bar{\rho}$ as given by (36) and (37), the eigenfunction ϕ and eigenvalue k_+ can easily be obtained by integrating (20) numerically. Here the eigenfunction is normalized according to (23). In dimensionless form the real and imaginary parts of $k_+\delta$ as functions of $\bar{\omega} = \Omega\delta/U$ are shown in figure 4 for various Mach numbers. To find the coupling constants F and G the quantity $(\partial\Delta/\partial k)_\Omega$ as required by (34) is computed by numerical differentiation. In figures 5 and 6 the real and imaginary parts of these constants are given in dimensionless form for $M = 0{\cdot}05$, $0{\cdot}2$, $0{\cdot}4$ and $0{\cdot}6$ over the entire unstable frequency range. To use these values the eigenfunction ϕ must be non-dimensionalized by $\rho_0 U^2$. As can

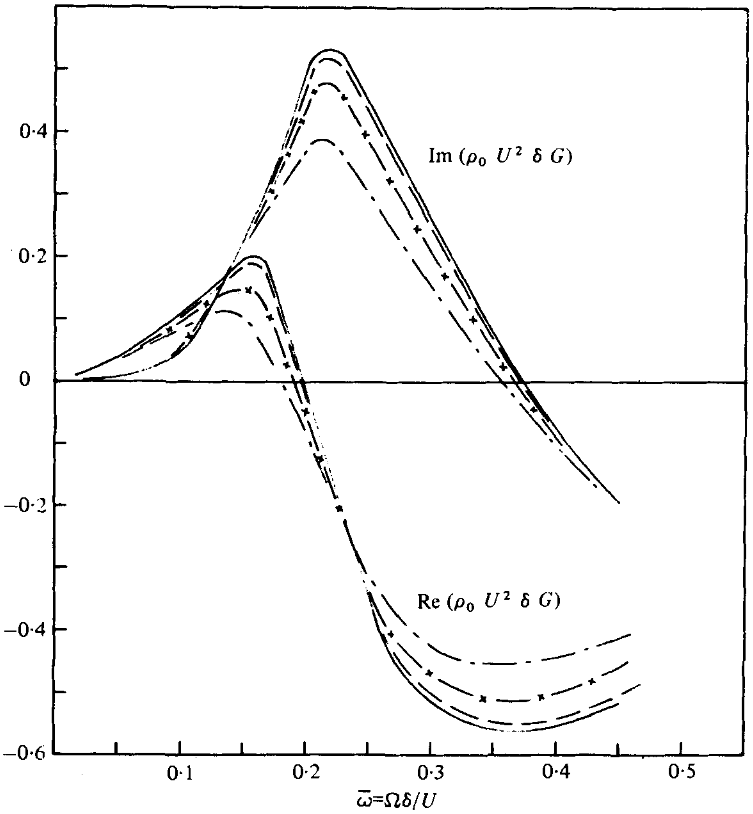

FIGURE 6. Real and imaginary parts of $\rho_0 U^2\delta G$ as a function of $\bar{\omega} = \Omega\delta/U$. ——, $M = 0.05$; --, $M = 0.2$; ×—×, $M = 0.4$; —·—, $M = 0.6$.

be seen from these figures the numerical values of $U\delta F$ and $\rho_0 U^2 \delta G$ are generally of the same order of magnitude. This implies that the excitation due to pressure fluctuations and velocity fluctuations of the incident sound wave are more or less of equal importance.

4. The effect of angle of incidence

Consider a narrow beam of sound waves incident on a two-dimensional shear layer. The effectiveness of the sound waves in exciting the instability waves of the shear layer obviously depends on the angle of incidence. We shall not attempt to study the effect of angle of incidence in the most general case. Instead we shall restrict our attention to the case in which the beam of sound waves has a spatial amplitude distribution more or less in the form of a Gaussian function. Let us take $g(k)$ in (3) and (4) to be of the form

$$g(k) = E\sigma \exp\left(-\tfrac{1}{4}\sigma^2(k-k_0)^2\right). \tag{38}$$

This amplitude function represents a beam of sound with horizontal wavenumber concentrated around $k = k_0$. The angle of incidence θ (see figure 7) measured from the x axis is related to k_0 by

$$k_0 = \frac{\Omega}{a_0}\cos\theta. \tag{39}$$

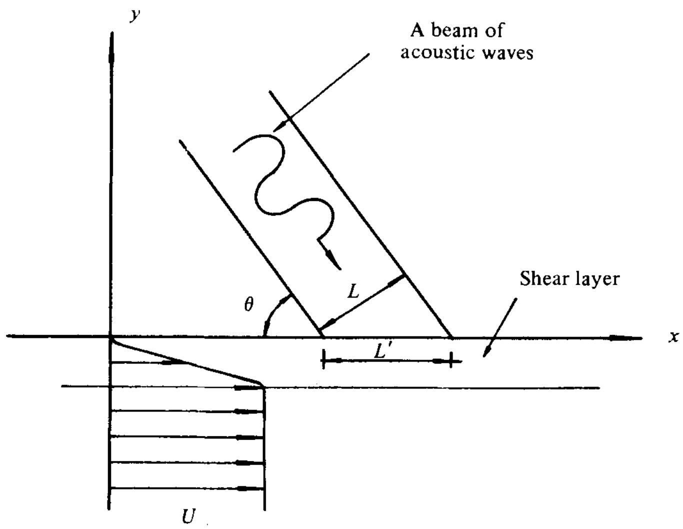

FIGURE 7. Schematic diagram of a beam of acoustic waves incident on a plane shear layer.

Substituting (38) into (3) we find that at the outer edge of the shear layer ($y = 0$) the pressure p_i is

$$p_i(x, 0, t) = 2\pi^{\frac{1}{2}} E \exp(-x^2/\sigma^2 + ik_0 x - i\Omega t). \tag{40}$$

The half-width of this pressure distribution L' is approximately equal to $1{\cdot}66\sigma$. From figure 7 it is seen that $L' \sin\theta \approx L$. Therefore, we have (L = half-width of the beam)

$$1{\cdot}66\sigma \sin\theta = L \quad \text{or} \quad \sigma = \frac{L}{1{\cdot}66 \sin\theta}. \tag{41}$$

Now downstream of the region of acoustic excitation the pressure associated with the instability wave is given approximately by (33) with the upper limit of the integral set equal to infinity:

$$p(x, y, t) = \left[G \int_{-\infty}^{\infty} \hat{p}_i(\xi, 0) \exp(-ik_+ \xi)\, d\xi + F \int_{-\infty}^{\infty} \hat{v}_i(\xi, 0) \exp(-ik_+ \xi)\, d\xi\right] \phi(y, k_+, \Omega) \times \exp(ik_+ x - i\Omega t)$$

$$= 2\pi\phi(y, k_+, \Omega) \exp(ik_+ x - i\Omega t)\left[G - \frac{F}{\rho_0 \Omega}(\Omega^2/a_0^2 - k_+^2)^{\frac{1}{2}}\right] g(k_+). \tag{42}$$

On using the expression for $g(k)$ given by (38) the last factor of the above expression can be put into a dimensionless form as follows:

$$\left[G - \frac{F}{\rho_0 \Omega}(\Omega^2/a_0^2 - k_+^2)^{\frac{1}{2}}\right] g(k_+) = \left[\rho U^2 \delta G - U\delta F\left(M^2 - \frac{(k_+\delta)^2}{\overline{\omega}^2}\right)^{\frac{1}{2}}\right] \frac{E\sigma \exp(-\frac{1}{4}\sigma^2(k_+ - k_0)^2)}{\rho U^2 \delta}, \tag{43}$$

where $\overline{\omega} = \Omega\delta/U$. For a beam of acoustic waves with a fixed maximum amplitude and a fixed half-width L, the angular dependence of the above expression comes only from the dependence of σ and k_0 on θ, namely, on the multiplication factor

$$\sigma \exp[-\tfrac{1}{4}\sigma^2(k_+ - k_0)^2].$$

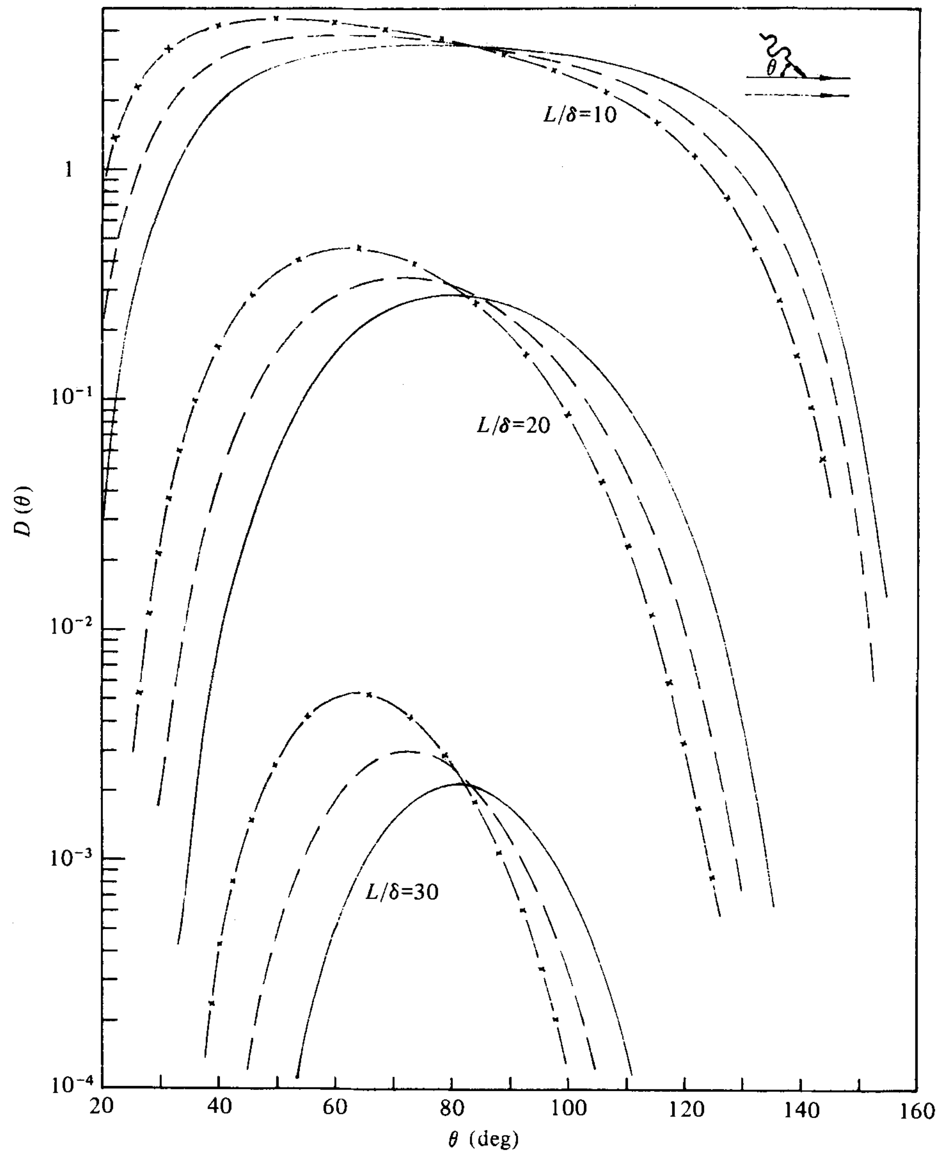

FIGURE 8. The directional sensitivity function $D(\theta)$. $\bar{\omega} = \Omega\delta/U = 0{\cdot}21$. ——, $M = 0{\cdot}2$; – –, $M = 0{\cdot}4$; ×—×, $M = 0{\cdot}6$.

This is proportional to,

$$\frac{L/\delta}{\sin\theta}\exp\left\{-\frac{L^2/\delta^2}{11{\cdot}02\sin^2\theta}(k_+\delta - M\bar{\omega}\cos\theta)^2\right\}. \tag{44}$$

The absolute value of this quantity, which we shall call the directional sensitivity function $D(\theta)$, is given by

$$D(\theta) = \frac{L/\delta}{\sin\theta}\left[\exp\left\{\frac{\operatorname{Re}(k_+\delta - M\bar{\omega}\cos\theta)^2}{\sin^2\theta}\right\}\right]^{-\frac{(L/\delta)^2}{11\cdot 02}}. \tag{45}$$

$D(\theta)$ is a measure of the relative sensitivity of the shear layer to sound induced instability as a function of the angle of incidence. In figure 8 the numerical values of $D(\theta)$ for $\bar{\omega} = 0{\cdot}21$ are shown. This value of $\bar{\omega}$ is practically the dimensionless frequency of the most unstable wave in the Mach number range of 0·0–0·6. Beams of three different sizes, namely $L/\delta = 10$, 20 and 30, are chosen. It is clear from this figure that a narrow beam is most effective in exciting instability waves. A broad beam with large L/δ is unlikely to have much effect in perturbing the shear layer. For the Mach number range considered, beams aimed at an angle of incidence between 50 and 80° will tend to generate instability waves with the largest amplitude. However, this angular dependence is more important for broad beams. The effectiveness of narrow beams is less sensitive to the angle of incidence. For a given beam the angle of incidence which is most effective in disturbing the shear layer decreases as the Mach number increases. Finally, we conclude from figure 8 that shear layers at higher Mach numbers are more sensitive to acoustic excitation.

To understand the above results physically it is important to recognize that for a given frequency effective coupling between the incident sound wave and the shear layer instability wave can be developed only if their wavenumbers are the same (or equivalently the phase velocities are the same). (*Note:* Here the matching of complex wavenumbers is involved. However, if the instability wave is only weakly unstable the interpretation of wavenumber to mean the real part of the wavenumber is essentially correct.) This is because unless the condition is satisfied the two-wave systems cannot be in phase with each other at all times. If they are out of phase any possible coupling would be cancelled out over the course of a period. In subsonic flows, however, the wavenumber of the shear layer instability wave is generally much larger than that of the incident acoustic wave. Thus these waves usually do not interact with each other unless the amplitude of the incident sound wave varies spatially. As a result of amplitude variation, the wavenumber spectrum of the incident sound wave need no longer be narrow and in fact could be very broad. When this happens there will be some wave components of the incident sound wave with a wavenumber which matches precisely that of the instability wave. These special wave components are responsible for the excitation of the unstable waves of the shear layer. Now if the incident sound wave is in the form of a narrow beam, its wavenumber spectrum would be very broad and overlap the wavenumber of the unstable wave of the shear layer. In this case strong excitation of the instability wave is to be expected. On the other hand if the beam of incident sound wave is very broad, its wavenumber spectrum would then be very narrow (a property of the Fourier transform) and centred around the wavenumber of the acoustic wave. Under these circumstances, the wavenumber matching condition may not be satisfied, resulting in negligibly small excited instability wave amplitude as shown in figure 8.

5. Summary

In this paper the receptivity problem of a plane shear layer to incident sound waves has been formulated mathematically. The excited instability wave solution is given. Numerical results indicate that excitations due to pressure fluctuations as well as velocity fluctuations associated with the incident sound waves are of more or less equal importance. To excite the instability wave of a shear layer, a narrow acoustic beam is most effective. For moderate subsonic Mach number the shear layer is most sensitive to acoustic beams aimed at an angle between 50 and 80° to the direction of flow.

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